

# Machta-Zwanzig regime of anomalous diffusion in infinite-horizon billiards

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We study diffusion on a periodic billiard table with infinite horizon in the limit of narrow corridors. An effective trapping mechanism emerges according to which the process can be modeled by a Lévy walk combining exponentially-distributed trapping times with free propagation along paths whose precise probabilities we compute. This description yields an approximation of the mean squared displacement of infinite-horizon billiards in terms of two transport coefficients which generalizes to this anomalous regime the Machta-Zwanzig approximation of normal diffusion in finite-horizon billiards [Phys. Rev. Lett. **50**, 1959 (1983)].

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The study of stochastic processes exhibiting anomalous transport has attracted considerable attention in the recent past, in particular in the context of continuous-time random walks [1–6]. It has since evolved into a highly interdisciplinary field, and includes a large array of applications [7].

From a fundamental perspective, a problem of specific interest concerns the emergence of such stochastic processes from deterministic dynamics [8, 9], and the derivation of their limiting distributions [10], which typically differ from Gaussian, as, for instance, Lévy stable distributions [11].

Infinite-horizon periodic billiard tables, which are the focus of this Letter, provide an example of mechanical systems whose transport properties exhibit a weak form of anomalous behavior, in the sense that there is a logarithmic correction to the linear growth in time of the mean squared displacement [12–14]; this is in fact a marginal case, separating diffusive and superdiffusive regimes [15]. It is known, in particular, that the distribution of the anomalously rescaled displacement vector,  $[\mathbf{r}(t) - \mathbf{r}(0)]/\sqrt{t \log t}$ , converges weakly to a normal distribution, whose covariance matrix is given in terms of a scalar expression of the model's parameter, the so-called Bleher variance [16–18].

As opposed to finite-horizon billiards, which are such that the displacement between two successive collisions of a point particle with obstacles remains bounded, infinite-horizon billiards admit, among their solutions, collisionless trajectories; point particles can therefore move arbitrarily far through infinite *corridors* devoid of obstacles. When the widths of these corridors is taken to be vanishingly small, a separation between two timescales occurs, the first associated with the propagation phases of a trajectory across the billiard table along the corridors and the second with an effective trapping mechanism which consists of a scattering phase that takes place within a single billiard cell, separating successive propagation phases. While the former timescale is independent of the widths of the corridors, the latter grows unbounded in the narrow-corridor regime.

The purpose of this Letter is to show that, in the limit of

narrow corridors, a novel regime emerges, such that a normal contribution to the finite-time diffusion coefficient may become arbitrarily larger than the anomalous component, even though the latter diverges logarithmically as time increases while the former remains asymptotically constant. Due to its connection with the Machta-Zwanzig approximation of the diffusion coefficient of finite-horizon billiard tables [19], which is obtained in a limit similar to that of narrow corridors, we refer to this limit as a Machta-Zwanzig regime of anomalous diffusion in infinite-horizon billiards.

To analyze this regime, we model the process on the infinite-horizon billiard table by a Lévy walk [20, 21]. Our approach is based on the framework of continuous-time random walks and relies on the distinction between the states of particles in propagating and scattering phases and the derivation of a multistate generalized master equation [22, 23]. The process whereby a walker on a lattice remains in a scattering phase during an exponentially-distributed waiting time and then propagates in a random direction over a free path of random length is such a multistate process and it can be solved analytically [24]. When applied to the infinite-horizon billiard, the probability distribution of free paths decays algebraically with the third power of their lengths [13, 16]. The mean squared displacement then has two relevant components: The first associated with the anomalous rescaling of the displacement vector [16–18], and the second with a normal diffusion component, similar to the Machta-Zwanzig approximation obtained in the finite-horizon regime [19]. Whereas the former contribution is second order in the widths of the corridors, the latter is first order and, for finite time, becomes much larger than the former.

We report numerical measurements of the mean square displacement of infinite-horizon periodic billiard tables which exhibit those two contributions, in agreement with the results predicted by the Lévy walk model.

**Model.** We consider the simplest kind of billiard in the plane, which is defined by a periodic array of square cells of sides  $\ell$  with identical circular scatterers of radii  $\rho$ ,  $0 <$

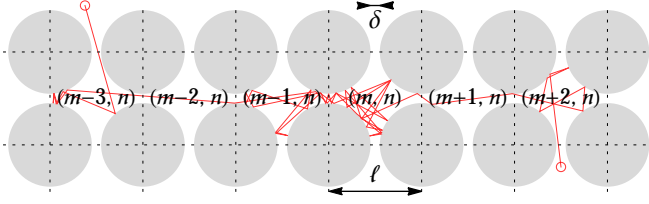


FIG. 1. (Color online) Infinite-horizon periodic billiard table of square cells of sides  $\ell$ . The discs centered at the cells' corners have radii  $\rho = 0.45\ell$ . Infinite corridors of widths  $\delta = \ell - 2\rho$  span along horizontal and vertical axes, through the centers of each cell. The broken red line shows a trajectory.

$\rho < \ell/2$ , whose centers are placed at the corners of the cells. A point-particle moves freely on the exterior of these obstacles, performing specular collisions upon their boundaries; see Fig. 1.

This is an *infinite-horizon* configuration, which refers to the existence of collisionless trajectories along the vertical and horizontal corridors that separate the obstacles. In contrast, a *finite-horizon* configuration would occur if, for instance, an additional circular obstacle of radius  $\rho_\odot$ , with  $\ell/2 - \rho < \rho_\odot < \ell/\sqrt{2} - \rho$ , were placed at the center of each cell [25], such that a particle would have to perform at least one collision in each visited cell. The widths of the corridors is  $\delta = \ell - 2\rho$ , which we take to be a control parameter.

Transport of particles on such billiard tables can be studied in terms of the time-evolution of the coarsegrained distribution of particles in cell  $\mathbf{n} \in \mathbb{Z}^2$ , whose changes are determined by the transitions particles make as they go from one cell to another. Generally speaking, this is a complicated process which is affected by correlations between successive collision events. The situation however simplifies when these correlations become negligible, which occurs when  $\delta \ll \ell$ .

*Infinite vs finite horizon.* In finite-horizon tables, where diffusion is always normal [26], this regime yields an approximation of the process by a continuous-time random walk for displacements on the lattice structure [27]. The associated timescale is given in terms of the residence time,

$$\tau_R = \frac{\pi A}{4p_0\delta}, \quad (1)$$

where  $A$  denotes the area of the elementary cell, *viz.*  $A = \ell^2 - \pi(\rho^2 + \rho_\odot^2)$ , and  $p_0$  is the speed of point particles [28].

When  $\tau_R$  is large with respect to the intercollisional time, a particle typically spends a long time rattling about the same cell, making many collisions with its obstacles, before reaching one of the four slits of widths  $\delta$ . Under these conditions, the velocity vectors of a particle entering and subsequently exiting a cell will effectively be uncorrelated. The motion of a particle on the billiard table can thus be approximated by a succession of scattering phases with random waiting times, exponentially-distributed with scale  $\tau_R$ , punctuated by instantaneous random hops from one cell to a neighboring one.

In this regime, the mean squared displacement grows linearly in time, with coefficient approximated by  $4D_{\text{MZ}}$ , where

$$D_{\text{MZ}} = \frac{\ell^2}{4\tau_R} \quad (2)$$

is a dimensional expression, known as the Machta-Zwanzig approximation to the diffusion coefficient [19]. To leading order, the actual diffusion coefficient differs from Eq. (2) by a correction which is expected to be linear in  $\delta/\ell$  [29]. The validity of the Machta-Zwanzig approximation thus relies on the separation  $\delta \ll \ell$ .

In the case of infinite horizon, the presence of corridors renders the Machta-Zwanzig regime more complex. In addition to the residence time  $\tau_R$ , Eq. (1), one has to take into consideration the timescale of propagation across a cell,  $\tau_F$ , which, in the narrow-corridor regime is small with respect to  $\tau_R$  [30],

$$\tau_F \equiv \frac{\ell}{p_0} \ll \tau_R. \quad (3)$$

When the phase-space coordinates of a particle are such that it crosses the boundary between two cells with velocity almost perpendicular to it, the next scattering phase may take place at a remote distance, after a time approximately equal to  $\tau_F$  multiplied by the number of cells traveled free of collisions.

The assumption that correlations between successive scattering phases become negligible amounts to approximating the transport process on the infinite-horizon billiard table by a Lévy walk, whose distribution of free paths, *i.e.* the distance a particle propagates free of collisions, matches that of the infinite-horizon billiard, taking into account the time-delay induced by the propagation along free paths. The general framework for analyzing Lévy walks where a distinction occurs between the states of particles in scattering and propagating phases was described in Ref. [24]. Whereas particles in a scattering state make transitions at random times, exponentially distributed with scale  $\tau_R$ , and move in a random direction to a neighboring cell, the transitions of particles in a propagating state are deterministic: They take place after exactly time  $\tau_F$  and are accompanied by displacements along the same direction as the previous transition.

*Parameter values.* We label the internal state of a walker by  $k \in \mathbb{Z}$ , with  $k = 0$  denoting the scattering state and  $k \geq 1$  a propagating state, where  $k$  corresponds to the total number of steps of duration  $\tau_F$  until the propagating state returns to a scattering one. The direction of propagation is fixed in the propagating state. When the particle is in the scattering state, we let  $\mu_k$  denote the probability of a transition to either a scattering state,  $k = 0$ , or a propagating state,  $k \geq 1$ , for any of the four lattice directions the walker can move in.

In the narrow-corridor regime, the overwhelming majority of transitions are from scattering to scattering states; only rarely do particles perform transitions from scattering to propagating states. When they do, however, as mentioned earlier, the probability of a long excursion to a propagating state,  $k \gg 1$ , decays with the third power of their lengths,  $\mu_k \sim k^{-3}$ .

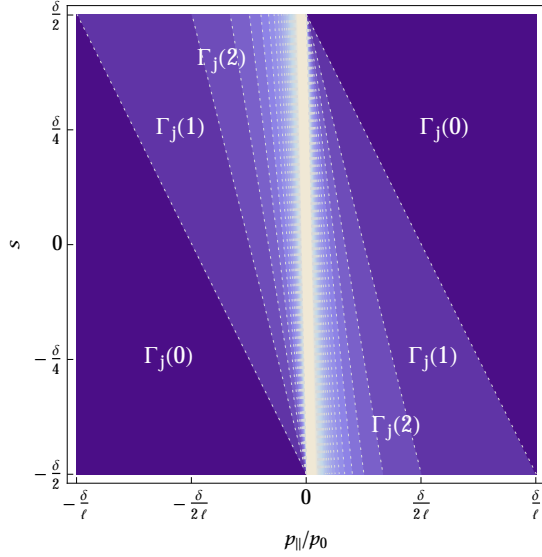


FIG. 2. (Color online) Graph showing two-dimensional phase-space sets according to the distance separating phase points at the boundary between two cells from their next collision on a disc. The darkest color encodes sets of points whose next collision takes place in the cell they move into; the brighter the colors the more distant the next collision. The two axes correspond, respectively, to  $p_{\parallel}/p_0$ , which denotes the fraction of the velocity component directed along the boundary,  $-1 < p_{\parallel}/p_0 < 1$  (here restricted to  $-\delta/\ell < p_{\parallel}/p_0 < \delta/\ell$ ) and to the position  $s$  along the boundary,  $-\delta/2 < s < \delta/2$ , where  $\delta = \ell - 2\rho$  ( $\rho = 0.45\ell$ ), is the width of the corridor. With these coordinates, the area corresponds to the natural invariant measure, which is normalized if divided by  $2\delta$ .

In fact, to first order in  $\delta/\ell$ , we can write

$$\mu_k = \begin{cases} 1 - \frac{\delta}{4\ell}, & k = 0, \\ \frac{\delta}{k(k+1)(k+2)\ell}, & k \geq 1. \end{cases} \quad (4)$$

To obtain this result, consider the phase-space sets that lie at the boundary between two neighboring cells  $\mathbf{n}$  and  $\mathbf{n} + \mathbf{e}_j$ , where  $\mathbf{e}_j$  is a unit vector in one of the four lattice directions,  $j \in \{1, \dots, 4\}$ . A particle in cell  $\mathbf{n}$  which crosses over to cell  $\mathbf{n} + \mathbf{e}_j$  is mapped at this boundary to a phase point with coordinate  $|s| < \delta/2$ , along the direction of the separation between the two cells, with velocity  $\mathbf{p}$  such that  $\mathbf{p} \cdot \mathbf{e}_j \equiv p_{\parallel} > 0$ .

Let  $\Gamma_j(k)$  denote the set of such phase points which are mapped by the flow to the next collision event on an obstacle in cell  $\mathbf{n} + k\mathbf{e}_j$ . Since the points of these sets can be mapped back to their preceding collision on an obstacle in cell  $\mathbf{n} - l\mathbf{e}_j$ , for some  $l \geq 0$ , they are associated with point particles in a propagation phase of length at least  $k$ . We must measure these points by means of the natural invariant measure, the one induced by the Liouville measure on the cross-sections defined by collisions or cell crossings, which is known to be the area in phase space as parameterized in Fig. 2. Therefore, up to a normalization factor, the area of  $\Gamma_j(k)$  is given by  $\sum_{i=k}^{\infty} \mu_i$ . By geometric arguments, and as can also be seen from Fig. 2, the measure of  $\cup_{i=k}^{\infty} \Gamma_j(i)$  is, to leading order in  $\delta/\ell$ , proportional

to the area of a right triangle of base  $\delta/(k\ell)$ . We therefore have, for  $k \geq 1$ ,

$$\sum_{i=k}^{\infty} \sum_{j=1}^4 \mu_j = \frac{\delta}{2k\ell}, \quad (5)$$

which implies, and thus justifies, Eq. (4).

*Anomalous diffusion.* We let  $\mathbf{r} = \mathbf{n}\ell$  denote the displacement of Lévy walkers on the two-dimensional square lattice, measured from the origin where they are initially located, and obtain an expression of their mean squared displacement as a function of time,  $\langle r^2 \rangle_t$ , in terms of the time-integral of the overall fraction of particles in a scattering state,  $\sigma(t)$ ,

$$\frac{\langle r^2 \rangle_t}{\ell^2} = \frac{1}{\tau_R} \int_0^t ds \sigma(s) + \frac{\delta}{2\ell\tau_R} \sum_{k=1}^{\lfloor t/\tau_F \rfloor} \frac{2k+1}{k(k+1)} \int_0^{t-k\tau_F} ds \sigma(s); \quad (6)$$

see Ref. [24] for further details. In the stationary state, the fraction of particles in a scattering state is simply given by the ratio of the average time spent in the scattering state,  $\tau_R$ , to the average return time to this state,  $\tau_R + \sum_{k=1}^{\infty} k\mu_k\tau_F$ . Substituting the transition probabilities, Eq. (4), yields

$$\lim_{t \rightarrow \infty} \sigma(t) = \frac{\tau_R}{\tau_R + \frac{\delta}{2\ell}\tau_F} \simeq 1 - \frac{\delta}{2\ell} \frac{\tau_F}{\tau_R}. \quad (7)$$

Plugging this expression into Eq. (6), we obtain an expression of the mean square displacement of walkers in terms of harmonic numbers, which, in the long-time limit, reduces to

$$\frac{\langle r^2 \rangle_t}{4t} = \frac{\ell^2}{4\tau_R} + \frac{\delta\ell}{4\tau_R} [\log t + \mathcal{O}(1)]. \quad (8)$$

This is our main result. It generalizes to infinite-horizon billiard tables in the narrow-corridor regime the Machta-Zwanzig approximation of the diffusion coefficient for finite-horizon tables, Eq. (2).

For short times, the right-hand side of Eq. (8) is dominated by the constant term, which, to leading order in  $\delta/\ell$ , corresponds to a regime of normal diffusion, with the coefficient (2), consistent with the Machta-Zwanzig approximation of the (normal) diffusion coefficient.

The term carrying the logarithmic correction has coefficient  $\delta\ell/(4\tau_R)$ , which is identical to the Bleher limiting variance of the anomalously rescaled limiting distribution [16–18], i.e. such that the displacement vector rescaled by the square root of  $t \log t$  converges in distribution to a centered normal distribution whose covariance matrix reduces to a scalar given by this coefficient [31].

Whereas the coefficient of the constant term on the right-hand side of Eq. (8) is first order in the small parameter  $\delta/\ell$ , the coefficient of the logarithmic term is second order. Such a contribution thus becomes significant only for times such that  $\log t \approx \ell/\delta$ . In the narrow-corridor regime, however, the constraints on the integration times are such that  $\log t$  remains small with respect  $\ell/\delta$ .

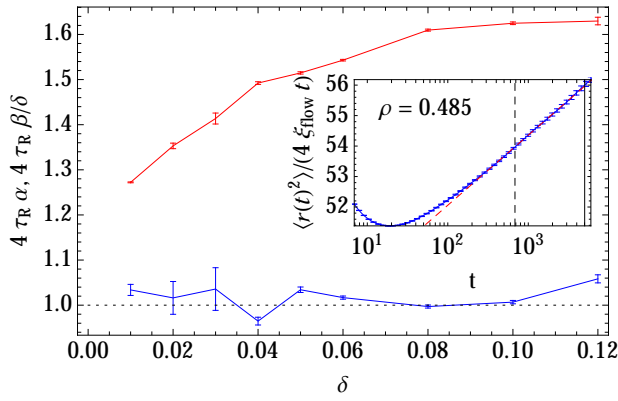


FIG. 3. (Color online) Computed slopes,  $\beta$  (blue, lower curve), and intercepts,  $\alpha$  (red, upper curve), of the mean squared displacement of point particles on the infinite-horizon Lorentz gas, fitted according to Eq. (9) for different values of the parameter  $\delta$ . The values found are here normalized by those predicted by Eqs. (10) and (11). To illustrate the fitting procedure, the inset shows the mean squared displacement (blue curve, including error bars) measured, as a function of time, for  $\delta = 0.03$ . The red dashed line is the result of a linear fit performed in the interval marked by the two vertical lines; see Ref. [32] for further details on this procedure. The units are chosen such that  $\ell \equiv p_0 \equiv 1$ .

*Numerical results.* Following the results presented in Ref. [32], we perform numerical measurements of the mean squared displacement of infinite-horizon billiard tables such as shown in Fig. 1 and determine a range of time values such that the distribution of free paths is accurately sampled, which, according to Eq. (4), scales like the square root of the total number of initial conditions taken (typically  $10^9$ ). In that range, we seek an asymptotically affine fitting function of  $\log t$  for the normally rescaled mean squared displacement,

$$\frac{\langle r^2 \rangle_t}{4t} \sim \alpha + \beta \log t, \quad (9)$$

where  $\alpha$  and  $\beta$  are implicit functions of time, and are expected to converge to the values predicted by Eq. (8) as  $t \rightarrow \infty$ , i.e.

$$\lim_{\delta \rightarrow 0} \lim_{t \rightarrow \infty} \frac{4\tau_R \alpha}{\ell^2} = 1, \quad (10)$$

$$\lim_{t \rightarrow \infty} \frac{4\tau_R \beta}{\delta \ell} = 1, \quad (11)$$

where, in the first line, the narrow-corridor limit takes care of  $\delta$ -dependent corrections to  $\alpha$  our theory does not account for.

Values found for these fitting parameters are reported in Fig. 3 for different values of  $\delta$ . For the parameter  $\beta$ , on the one hand, one expects Eq. (11) to hold for all values of  $\delta$  in the range shown and, in view of the difficulties presented by such measurements [32], the agreement is indeed rather good, especially given the prevalence of finite-time effects when  $\delta \rightarrow 0$ . There is, on the other hand, no analytic prediction for the value of the parameter  $\alpha$ , other than that given in the narrow-corridor limit, Eq. (10). Nevertheless, our data provides clear evidence in support of this result.

We should note that, in contrast to the approximating Lévy walk, for which corrections to the zeroth order result (8) are found to be negative, the corrections to the zeroth order result for measurements performed for billiards appear to be positive at first order in  $\delta/\ell$ . This is to be expected, since memory effects should indeed bring about corrections of the same order, as is the case with finite-horizon billiards [29]; such corrections may well predominate.

*Conclusions* Infinite-horizon billiard tables with narrow corridors display anomalous transport properties such that the logarithmic divergence in time of the mean squared displacement must effectively be treated as a subleading contribution with respect to a normally diffusive one.

Our stochastic analysis of the process in terms of a Lévy walk with both scattering phases, characterized by random waiting times with exponential distributions, and propagating phases along the table's corridors provides two quantitative predictions which match the Machta-Zwanzig dimensional prediction of the (normal) diffusion coefficient, on the one hand, and the Bleher variance of the anomalously rescaled process, on the other hand. Their physical interpretations is, moreover, transparent: (i) the overwhelming majority of transitions taking place on the billiard table are similar to those observed in finite-horizon billiard tables, giving rise to the predominant normal contribution to the mean squared displacement, and (ii) rare scattering events allow propagation along the billiard's corridors over long distances whose lengths follow a precise distribution, at the origin of the anomalous contribution to the mean squared displacement. As our numerical results make clear, ignoring the first of these two contributions would obstruct the accurate measurement of the second.

We conclude by observing that the scaling properties of the transition probabilities, Eq. (4), can be generalized to other values, extending the relevance of our approach well beyond the regime studied in this Letter. As discussed in Ref. [24], tuning the parameter values allows the description of both normal and anomalous transport regimes, including ballistic transport. Further applications will be reported elsewhere.

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- [1] J. W. Haus and K. W. Kehr, *Physics Reports* **150**, 263 (1987).
  - [2] J.-P. Bouchaud and A. Georges, *Physics Reports* **195**, 127 (1990).
  - [3] G. H. Weiss, *Aspects and Applications of the Random Walk* (North-Holland, Amsterdam, 1994).
  - [4] M. F. Shlesinger, G. M. Zaslavsky, and U. Frisch, eds., *Lévy Flights and Related Topics in Physics*, Lecture Notes in Physics, Vol. 450 (Springer, Berlin, Heidelberg, 1995).
  - [5] S. Benkadda and G. M. Zaslavsky, eds., *Chaos, Kinetics and*



*Nonlinear Dynamics in Fluids and Plasmas*, Lecture Notes in Physics, Vol. 511 (Springer, Berlin Heidelberg, 1998).

- [6] G. M. Zaslavsky, *Physics Reports* **371**, 461 (2002).
- [7] R. Klages, G. Radons, and I. M. Sokolov, *Anomalous transport: Foundations and applications* (Wiley-VCH Verlag, Weinheim, 2008).
- [8] M. F. Shlesinger, G. M. Zaslavsky, and J. Klafter, *Nature* **363**, 31 (1993).
- [9] G. M. Zaslavsky, *Physics Today* **52**, 39 (1999).
- [10] J. Klafter, M. F. Shlesinger, and G. Zumofen, *Physics Today* **49**, 33 (1996).
- [11] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, revised ed. (Addison-Wesley, New York, 1968).
- [12] B. Friedman and R. F. Martin, Jr., *Physics Letters A* **105**, 23 (1984).
- [13] J.-P. Bouchaud and P. Le Doussal, *Journal of Statistical Physics* **41**, 225 (1985).
- [14] A. Zacherl, T. Geisel, J. Nierwetberg, and G. Radons, *Physics Letters A* **114**, 317 (1986).
- [15] T. Geisel, J. Nierwetberg, and A. Zacherl, *Physical Review Letters* **54**, 616 (1985).
- [16] P. M. Bleher, *Journal of Statistical Physics* **66**, 315 (1992).
- [17] D. Szász and T. Varjú, *Journal of Statistical Physics* **129**, 59 (2007).
- [18] D. I. Dolgopyat and N. I. Chernov, *Russian Mathematical Surveys* **64**, 651 (2009).
- [19] J. Machta and R. Zwanzig, *Physical Review Letters* **50**, 1959 (1983).
- [20] J. Klafter, A. Blumen, and M. F. Shlesinger, *Physical Review A* **35**, 3081 (1987).
- [21] G. Zumofen and J. Klafter, *Physical Review E* **47**, 851 (1993).
- [22] V. M. Kenkre, E. W. Montroll, and M. F. Shlesinger, *Journal of Statistical Physics* **9**, 45 (1973).
- [23] U. Landman, E. W. Montroll, and M. F. Shlesinger, *Proceedings of the National Academy of Sciences of the United States of America* **74**, 430 (1977).
- [24] G. Cristadoro, T. Gilbert, M. Lenci, and D. P. Sanders, *arXiv:1407.0227* (2014).
- [25] Without loss of generality, we assume  $\rho \geq \rho_\odot$ . For definiteness of a unique relevant timescale, Eq. (1), the separation between the central disc and the outer ones should also be substantially larger than the separation between two outer discs.
- [26] L. A. Bunimovich, Y. G. Sinai, and N. Chernov, *Russian Mathematical Surveys* **46**, 47 (1991).
- [27] P. Gaspard and T. Gilbert, *Chaos* **22**, 026117 (2012).
- [28] N. Chernov, *Journal of Statistical Physics* **88**, 1 (1997).
- [29] T. Gilbert and D. P. Sanders, *Physical Review E* **80**, 041121 (2009).
- [30] Strictly speaking, the residence time, Eq. (1), accounts for the possibility of collisionless motion inside a cell. However in the narrow-corridor regime, the difference between  $\tau_R$  and the actual residence time conditioned on particles performing collisions inside the cell is next order in the small parameter  $\delta/\ell$  and will thus be neglected.
- [31] The second moment actually scales with coefficient given by twice the Gaussian value. See discussion in Ref. [32].
- [32] G. Cristadoro, T. Gilbert, M. Lenci, and D. P. Sanders, *Physical Review E*, in press; *arXiv:1405.0975* (2014).